Torsion groups and the Bienvenu–Geroldinger conjecture

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Outline

1. **Power semigroups and the Tamura–Shafer problem**
2. The Bienvenu–Geroldinger problem
3. Preliminaries on the order of an element in a monoid
4. The two-to-two property
5. Torsion groups and the Bienvenu–Geroldinger conjecture
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Power semigroups and power monoids

Herein, all sgrps are written multiplicatively unless stated otherwise.

The large power semigroup of a sgrp S is the sgrp $\mathcal{P}(S)$ obtained by endowing the *non-empty* subsets of S with the (provably associative) operation

$$(X, Y) \mapsto XY := \{xy \colon x \in X,\ y \in Y\}.$$

The finitary power semigroup of a sgrp S is the subsemigroup of $\mathcal{P}(S)$ consisting of all finite *non-empty* subsets of S.
If M is a monoid with identity 1_M, then $\mathcal{P}(M)$ is itself a monoid with identity $\{1_M\}$ and is therefore called the large power monoid of M.

Each of the following is a submonoid of $\mathcal{P}(M)$:

- $\mathcal{P}_{\mathrm{fin}}(M) := \{X \in \mathcal{P}(M) \colon |X| < \infty\}$, the finitary power monoid of M.
- $\mathcal{P}_{\mathrm{fin},1}(M) := \{X \in \mathcal{P}_{\mathrm{fin}}(M) \colon 1_M \in X\}$, the reduced finitary power monoid of M.

Altogether, these structures will be referred to as power monoids and power semigroups.

Global isomorphisms

A milestone in the study of power sgrps was marked by a 1967 paper of Tamura and Shafer(2) that has eventually led to the following questions.

Questions 1.

Let $\mathscr{C}$ be a class of sgrps. Given $H, K \in \mathscr{C}$, is it true that

(1) $\mathcal{P}(H)$ is isomorphic to $\mathcal{P}(K)$ if and only if H is isomorphic to K?

(2) $\mathcal{P}_{\text{fin}}(H)$ is isomorphic to $\mathcal{P}_{\text{fin}}(K)$ if and only if H is isomorphic to K?

Both questions are special cases of the following (very general) problem:

Question 2 (Functorial isomorphism problem).

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a class $\mathcal{O}$ of objects of $\mathcal{C}$, prove or disprove that $F(A) \cong_{\mathcal{D}} F(B)$, for arbitrary $A, B \in \mathcal{O}$, iff $A \cong_{\mathcal{D}} B$, where $\cong_{\mathcal{C}}$ (resp., $\cong_{\mathcal{D}}$) implies an isomorphism in $\mathcal{C}$ (resp., in $\mathcal{D}$).

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(2) T. Tamura and J. Shafer, *Power semigroups*, Math. Japon. **12** (1967), 25–32.

The Tamura–Shafer problem

Question 1(1) is known as the Tamura–Shafer problem and was quickly answered in the negative for *arbitrary* sgrps(3), but has a positive answer in many significant cases (the list is *not* complete):

- *finite* groups and *finite* chains, see Theorems 5.8 and 5.9 in [Tamura & Shafer, Math. Japon. **12** (1967), 25–32].
- groups [Shafer, Math. Japon. **12** (1967), 32].
- *unital* semilattices, chains, and lattices, see Theorems 1.3, 1.4, and 2.2 in [Gould, Iskra, & Tsinakis, Algebra Univ. **19** (1984), 137–141].
- *finite* simple sgrps and *finite* semilattices of torsion groups, see Theorems 3.3 and 2.2 in [Gould & Iskra, Semigroup Forum **28** (1984), 1–11].
- semilattices, see p. 218 in [Kobayashi, Semigroup Forum **29** (1984), 217–222].
- completely 0-simple sgrps and completely simple sgrps, see Theorems 5.9 and 6.8 in [Tamura, J. Algebra **98** (1986), 319–361].
- Clifford sgrps, see Theorem 4.7 in [Gan & Zhao, J. Aust. Math. Soc. **97** (2014), 63–77].

The problem is open, e.g., for *finite* sgrps and *cancellative*(4) sgrps, but was solved in the cancellative *commutative* setting, both in its original form and in the variant for *finitary* power sgrps [Tringali, 2024]. The latter result has been further generalized to the cancellative *duo* setting [Li & Tringali, 202?].

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(3) See [Mogiljanskaja, Semigroup Forum **6** (1973), 330–333].

(4) A sgrp S is cancellative if $x \mapsto ax$ and $x \mapsto xa$ are injective fncs on S for every $a \in S$.

Recent literature and popularization

Power sgrps went dormant and were rediscovered in 2018:

- Fan & Tringali, J. Algebra **512** (2018), 252–294.

The paper brought new life to the topic and has been followed by a few more:

- Antoniou & Tringali, Pacific J. Math. **312** (2021), No. 2, 279–308.
- Sect. 4.2 in Tringali, J. Algebra **602** (July 2022), 352–380.
- pp. 101–102 in Geroldinger & Khadam, Ark. Mat. **60** (2022), 67–106.
- Bienvenu & Geroldinger, Israel J. Math. **265** (2025), 867–900.
- Example 4.5(3) and Remark 5.5 in Cossu & Tringali, J. Algebra **630** (2023), 128–161.
- Tringali & Yan, Proc. Amer. Math. Soc. **153** (2025), No. 3, 913–919.
- Gonzalez et al., Intl. J. Algebra Comput. **35** (2025), No. 2, 167–181.
- Tringali & Yan, J. Comb. Theory Ser. A **209** (2025), #105961, 16 pp.
- García-Sánchez & Tringali, Proc. Amer. Math. Soc. (2025), No. 6, 2323–2339.
- [Preprints] Cossu & Tringali (arXiv:2503.08615), Aggarwal et al. (arXiv:2412.05857), Gotti et al. (arXiv:2501.03407), Wen & Tringali (arXiv:2504.12566), etc.

In 2023, power sgrps were the subject of a CrowdMath project led by F. Gotti:

`https://artofproblemsolving.com/polymath/mitprimes2023`

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The Bienvenu–Geroldinger conjecture

More recently, Bienvenu and Geroldinger**(5)** considered another instance of the Functorial Isomorphism Problem (Question 2), which has led them to the formulation of the following conjecture:

Bienvenu–Geroldinger conjecture

The reduced finitary PM of a numerical monoid S is isomorphic to the reduced finitary PM of a numerical monoid T iff $S = T$.

Since two numerical monoids are equal iff they are isomorphic, the conjecture has eventually prompted me and Tringali to propose the next problem, which serves as a natural complement to Questions 1:

Question 3.

Let $\mathscr{C}$ be a class of monoids. Given $H, K \in \mathscr{C}$, is it true that $\mathcal{P}_{\mathrm{fin},1}(H)$ is isomorphic to $\mathcal{P}_{\mathrm{fin},1}(K)$ if and only if H is isomorphic to K?

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(5) P.-Y. Bienvenu and A. Geroldinger, *On algebraic properties of power monoids of numerical monoids*, Israel J. Math. 265 (2025), 867–900.

Solutions and new directions

In general, Question 3 has a negative answer, as shown by the following:

Let H be a 2-element monoid. Up to isomorphism, H is then either the additive group of integers modulo 2, or the idempotent submonoid $E = \{0, 1\}$ of the multiplicative monoid of the ring of integers. In both cases, $\mathcal{P}_{\mathrm{fin},1}(H)$ is an idempotent 2-element monoid (whose elements are the sets $\{1_H\}$ and H), and hence it is isomorphic to E.

However, one can prove the following theorem, which, in particular, confirms the Bienvenu–Geroldinger conjecture:

Theorem 1 (Tringali and Yan · Proc. AMS, 2025).

Question 3 has a positive answer for the class of Puiseux monoids, that is, submonoids of the non-negative rational numbers under addition.

It is natural to ask whether the theorem holds for other classes of monoids. Recently, Rago has first extended Theorem 1 to arbitrary groups and then completely characterized the pairs (H, K) of cancellative monoids for which Question 3 has a positive answer.

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Detecting the order via set powers

This talk focuses on the class of torsion groups, that is, groups in which every element has finite order. Suppose now that H is a monoid. The order $\mathrm{ord_H(a)}$ of an element $a \in H$ is the *size* of the cyclic submonoid $\langle a \rangle_H := \{a^n : n \in \mathbb{N}\}$ generated by a: if $\langle a \rangle_H$ is infinite, then $\mathrm{ord_H(a)} := \infty$; otherwise, $\mathrm{ord_H(a)}$ is the number of elements in $\langle a \rangle_H$.

Lemma 2.

Let H be a monoid. A torsion element $z \in H$ has order n if and only if n is the smallest integer $k \geq 1$ such that $\{1_H, z\}^k = \{1_H, z\}^{k-1}$.

Proof.

Fix $z \in H$. z has order n if and only if $1_H, z, \ldots, z^{n-1}$ are pairwise distinct and there exists $m \in [\![0, n-1]\!]$ such that $z^n = z^m$. This implies, on the one hand, that

$$\{1_H, z\}^n = \{1_H, z, \ldots, z^n\} = \{1_H, z, \ldots, z^{n-1}\} = \{1_H, z\}^{n-1}; \qquad (1)$$

and, on the other hand, that

$$z^k \notin \{1_H, z, \ldots, z^{k-1}\} = \{1_H, z\}^{k-1}, \qquad \text{for all } k \in [\![1, n-1]\!]. \qquad (2)$$

In particular, Eq. (2) shows that $\{1_H, z\}^k \neq \{1_H, z\}^{k-1}$ for every $k \in [\![1, n-1]\!]$, which, together with Eq. (1), is enough to complete the proof. ■

Property involving order

Lemma 3.

Let H be a monoid, and fix $z \in H$ and $\ell \in \mathbb{N}^{+}$. If $r \geq \ell - 1$ is an integer, then

$$\{1_H, z^{\ell}\}\{1_H, z\}^{r} = \{1_H, z\}^{\ell + r}.$$

If, in addition, z is cancellative and ℓ is no larger than the order of z in H, then

$$\{1_H, z^{\ell}\}\{1_H, z\}^{r} \neq \{1_H, z\}^{s}, \qquad \text{for all } r, s \in \mathbb{N} \text{ with } r < \ell - 1 \leq s.$$

Proof.

The first part is straightforward: for any integer $r \geq \ell - 1$, we have $\{1_H, z^{\ell}\}\{1_H, z\}^{r} = \{z^{i} : 0 \leq i \leq r\} \cup \{z^{\ell+i} : 0 \leq i \leq r\} = \{1_H, z, \ldots, z^{\ell+r}\} = \{1_H, z\}^{\ell+r}$.
As for the second part, suppose that z is a cancellative element and ℓ is no larger than $n := \mathrm{ord_H(z)}$. Assume for a contradiction that there exist $r, s \in \mathbb{N}$ with $r < \ell - 1 \leq s$ such that

$$\{1_H, z\}^{s} = \{1_H, z^{\ell}\}\{1_H, z\}^{r} = \{1_H, z\}^{r} \cup z^{\ell}\{1_H, z\}^{r}. \tag{3}$$

Since $1 \leq \ell \leq n$, it is then clear that $z^{\ell-1} \notin \{1_H, z\}^{r}$ and $z^{\ell-1} \in \{1_H, z\}^{s}$. It thus follows from Eq. (3) that $z^{\ell-1} = z^{\ell+i}$ for some $i \in [\![0, r]\!]$. By the cancellativity of z, this implies $z^{i+1} = 1_H$. In particular, z is a unit and generates a finite cyclic subgroup of the unit group of H of order n. Therefore, we conclude from the elementary properties of cyclic groups that $n \mid i + 1$, which is impossible since $1 \leq i + 1 \leq r + 1 \leq \ell - 1 < n$. ■

Two equations

Lemma 4.

Let H be a monoid. If $x, y \in H$ and $x^r = y^s$ for some $r, s \in \mathbb{N}^+$, then

$$\{1_H, x\}^{r-1}\{1_H, xy\}\{1_H, y\}^s = \{1_H, x\}^r\{1_H, y\}^{s+1} \tag{4}$$

and

$$\{1_H, x\}^r\{1_H, xy\}\{1_H, y\}^{s-1} = \{1_H, x\}^{r+1}\{1_H, y\}^s. \tag{5}$$

Proof.

We focus on the first identity, as the second follows by symmetry.
Denote by L and R the left-hand side and the right-hand side of Eq. (4), respectively. Given $z \in H$, it is clear that $z \in L$ if and only if either $z = x^i y^j$ or $z = x^{i+1}y^{j+1}$ for some $i \in [\![0, r-1]\!]$ and $j \in [\![0, s]\!]$, while $z \in R$ if and only if $z = x^i y^j$ for some $i \in [\![0, r]\!]$ and $j \in [\![0, s+1]\!]$. Consequently, we have

$$R \smallsetminus \{x^r, y^{s+1}\} \subseteq L \subseteq R.$$

It remains to check that $x^r, y^{s+1} \in L$, and this is immediate. Indeed, we have $x^r = y^s$ (by hypothesis), and hence $y^{s+1} = x^r y$. Since $y^s, x^r y \in L$ (as noted above), we are done. ■

Relations between torsion elements

Theorem 4.

Let H be a monoid, and let $x, y \in H$ be cancellative torsion elements. Then x and y are units, and there exist $r, u \in [\![1, \mathrm{ord_H(x)}]\!]$ and $s, v \in [\![1, \mathrm{ord_H(y)}]\!]$ such that

- $x^r = y^s$ and $x^u = y^v$;
- If $x^c = y^d$ for some $c, d \in \mathbb{Z},$ then $r \mid c$ and $v \mid d$.

This result gives us a precise language to describe how two torsion elements are related through their powers. It will be essential when we later analyze the behavior of the pullback map on products like xy, as it provides the minimal relations we can rely on.

Outline

The two-to-two property

Lemma 5.

Let H be a monoid, S a subset of H containing the identity 1_H, and n an integer ≥ 3. If $T \subseteq S$ and $1_H \notin T$, then $(S^{n-1} \smallsetminus T)S = S^n$. In particular, the equation $AS = S^n$ has at least $2^{|S|-1}$ solutions $A \in \mathcal{P}(H)$ such that $1_H \in A$, and each of these solutions is finite whenever S is.

Proof.

Let T be a subset of $S \smallsetminus \{1_H\}$, and define $Q := S^{n-1} \smallsetminus T$. It is obvious that $QS \subseteq S^{n-1}S = S^n$, so we only need to check that $S^n \subseteq QS$. To this end, fix $z \in S^n$, and let k be the smallest non-negative integer such that $z \in S^k$. Our goal is to show that $z \in QS$, and we distinguish three cases.

- Case 1: $k = 0$ or $k = 1$. If $k = 0$, then $z \in S^0 = \{1_H\}$ and thus $z = 1_H$. It follows that $z \in S$ and hence $z \in 1_H S \subseteq QS$ (note that $1_H \in Q$).
- Case 2: $2 \leq k \leq n - 1$. We have $z \notin T$, or else $z \in S$, contradicting the definition itself of k. It follows that $z \in S^k \smallsetminus T \subseteq Q$, and hence $z \in Q1_H \subseteq QS$.
- Case 3: $k = n$. By the definition of k, we have that $z \notin S^i$ for each $i \in [\![1, n-1]\!]$. On the other hand, $z = s_1 \cdots s_n$ for some $s_1, \ldots, s_n \in S$. It follows that $z' := s_1 \cdots s_{n-1} \notin T$, or else $z = z's_n \in S^2$, which is a contradiction because $2 < 3 \leq n$. Therefore, $z' \in S^{n-1} \smallsetminus T$ and hence $z = z's_n \in QS$.

If T_1 and T_2 are distinct subsets of $S \smallsetminus \{1_H\}$, then the sets $S^{n-1} \smallsetminus T_1$ and $S^{n-1} \smallsetminus T_2$ are likewise distinct. As a result, the equation $AS = S^n$ has at least as many solutions $A \in \mathcal{P}(H)$ with $1_H \in A$ as there are subsets of $S \smallsetminus \{1_H\}$; namely, at least $2^{|S|-1}$ solutions. ■

The two-to-two property

The remarkable abundance of solutions revealed by Lemma 5 gives us a powerful counting tool, which we will use to prove a fundamental structural property of isomorphisms between power monoids.

Theorem 6.

Let f be an isomorphism from $\mathcal{P}_{\text{fin},1}(H)$ to $\mathcal{P}_{\text{fin},1}(K)$, where H and K are arbitrary monoids. Then $f(X)$ is a 2-element set for every 2-element set $X \in \mathcal{P}_{\text{fin},1}(H)$.

This is our first major structural result. It proves that any isomorphism between reduced finitary power monoids must send a 2-element set of the form $\{1, x\}$ to another 2-element set. The proof cleverly uses Lemma 5 to compare the number of solutions to a specific equation, forcing the sizes of the images to be the same.

The pullback

Since the image of $\{1, x\}$ is a 2-element set, we can now uniquely identify the element it pairs with the identity. This allows us to define a map between the original monoids themselves.

Corollary 7.

Let f be an isomorphism from $\mathcal{P}_{\text{fin},1}(H)$ to $\mathcal{P}_{\text{fin},1}(K)$, where H and K are arbitrary monoids. There is then a (uniquely determined) bijection $g\colon H \to K$ such that $f(\{1_H, x\}) = \{1_K, g(x)\}$ for each $x \in H$. In particular, $g(1_H) = 1_K$.

It remains an open question whether Theorem 6 can be generalized to show that, for arbitrary monoids H and K, every isomorphism from $\mathcal{P}_{\text{fin},1}(H)$ to $\mathcal{P}_{\text{fin},1}(K)$ is cardinality-preserving (the theorem yields a positive answer for sets of cardinality one or two); this remains unclear even in the cancellative case. Nevertheless, the result provides a far-reaching generalization of Theorem 1 and motivates the following definition, which plays a crucial role in this work.

Definition 8.

Let H and K be monoids. Based on Corollary 7, we let the pullback of an isomorphism $f\colon \mathcal{P}_{\text{fin},1}(H) \to \mathcal{P}_{\text{fin},1}(K)$ be the unique bijection $H \to K$ such that $f(\{1_H, x\}) = \{1_K, g(x)\}$ for all $x \in H$.

Outline

Properties of the pullback

With a concrete bijection g between the elements of H and K in hand, we can now start investigating its algebraic properties. The first and most natural question is whether it preserves the order of an element.

Theorem 9.

Let g be the pullback of an isomorphism f from $\mathcal{P}_{\mathrm{fin},1}(H)$ to $\mathcal{P}_{\mathrm{fin},1}(K)$, where H and K are arbitrary monoids. Then $\mathrm{ord}_H(x) = \mathrm{ord}_K(g(x))$ for every $x \in H$.

Proof.

Fix an element $x \in H$, and denote by n its order. Our objective is to show that $m := \mathrm{ord}_K(g(x)) = n$. We distinguish two cases, depending on whether n is finite or not.

Case 1: $n < \infty$. By Lemma 2, n is the smallest integer $k \geq 1$ such that $\{1_H, x\}^k = \{1_H, x\}^{k-1}$. Since, for all $X \in \mathcal{P}_{\mathrm{fin},1}(H)$ and $r \in \mathbb{N}$, the equality $X^r = X^{r-1}$ holds if and only if $f(X)^r = f(X)^{r-1}$, it follows that n is also the smallest integer $k \geq 1$ for which $\{1_K, g(x)\}^k = f(\{1_H, x\})^k = f(\{1_H, x\})^{k-1} = \{1_K, g(x)\}^{k-1}$. Therefore, by reapplying Lemma 2, we conclude that $m = n$.

Case 2: $n = \infty$. Given $i, j \in \mathbb{N}$ with $i < j$, we infer from the basic properties of cyclic semigroups that $x^i \neq x^j$. This in turn implies $\{1_H, x\}^i \subsetneq \{1_H, x\}^j$, and consequently $\{1_K, g(x)\}^i = f\big(\{1_H, x\}^i\big) \neq f\big(\{1_H, x\}^j\big) = \{1_K, g(x)\}^j$. It follows that $m = \infty = n$, completing the proof. ■

Properties of the pullback

Lemma 10.

Let g be the pullback of an isomorphism $f\colon \mathcal{P}_{\mathrm{fin},1}(H) \to \mathcal{P}_{\mathrm{fin},1}(K)$, where H and K are monoids with K cancellative. If $x \in H$ and $k \in \mathbb{N}$, then $g(x^k) = g(x)^\ell$ for some $\ell \in \mathbb{N}$ with $\ell \leq k$.

Theorem 11.

Let g be the pullback of an isomorphism f from $\mathcal{P}_{\mathrm{fin},1}(H)$ to $\mathcal{P}_{\mathrm{fin},1}(K)$, where H and K are cancellative monoids. Then $g(x^k) = g(x)^k$ for all $x \in H$ and $k \in \mathbb{N}$.

This theorem strengthens the previous result significantly. In cancellative monoids, it proves that the pullback g respects powers perfectly: $g(x^k) = g(x)^k$ for all elements x and all non-negative integers k. The proof uses the bijectivity of g and the bound from Lemma 10 to force the exponents to match.

Properties of the pullback

Having established that g behaves perfectly on powers of a single element, we now turn to the much more challenging problem of how it handles the product of two different elements, starting with the special case of torsion elements.

Lemma 12.

Let g be the pullback of an isomorphism f from $\mathcal{P}_{\mathrm{fin},1}(H)$ to $\mathcal{P}_{\mathrm{fin},1}(K)$, where H and K are cancellative monoids. If $x, y \in H$ are torsion elements, then either $g(xy) = g(x)g(y)$ or $x^2y^2 = 1_H$.

Set $m := \mathrm{ord_H(x)}$ and $n := \mathrm{ord_H(y)}$, and let $a := g(x)$ and $b := g(y)$. We know from Proposition 4 that $x, y \in H^\times$, and there exist $r, u \in [\![1, m]\!]$ and $s, v \in [\![1, n]\!]$ such that

$$\text{(i) } x^r = y^s \text{ and } x^u = y^v; \qquad \text{(ii) if } x^c = y^d \text{ for some } c, d \in \mathbb{Z}, \text{ then } r \mid c \text{ and } v \mid d. \tag{6}$$

We henceforth assume that $g(xy) \neq g(x)g(y)$, that is, $g(xy) \neq ab$; otherwise, there is nothing to do. To facilitate the presentation, the remainder of the proof is structured as a sequence of claims:
Claim A: $y \neq x^c$ and $x \neq y^d$ for all $c, d \in \mathbb{Z}$.
Claim B: $r \geq 2$ and $v \geq 2$.
Claim C: There exist $j, \ell \in \mathbb{N}$ with $2 \leq j < n$ and $2 \leq \ell < m$ such that $g(xy) = ab^j = a^\ell b$.
Claim D: $s \leq n - 2$ and $u \leq m - 2$.
Claim E: $s = n - 2$ and $u = m - 2$.
We have $x^r = y^{n-2} = y^{-2}$ and $y^v = x^{m-2} = x^{-2}$. Therefore, $x^2 = y^{-v}$ and $y^2 = x^{-r}$. This shows that $r = v = 2$. As a result, $x^2 = y^{-2}$, that is, $x^2y^2 = 1_H$.

Properties of the pullback

Our strategy is now clear: we must systematically eliminate the second, exceptional possibility. We begin by considering a special case where one of the elements has order 2.

Lemma 13.

Let g be the pullback of an isomorphism f from $\mathcal{P}_{\mathrm{fin},1}(H)$ to $\mathcal{P}_{\mathrm{fin},1}(K)$, where H and K are cancellative monoids. If $x, y \in H$ are torsion elements with $x^2 = 1_H$ or $y^2 = 1_H$, then $g(xy) = g(x)g(y)$.

Proof.

Assume to the contrary that $g(xy) \neq ab$, where $a := g(x)$ and $b := g(y)$. By Lemma 12, we have $x^2y^2 = 1_H$; and by the hypothesis that $x^2 = 1_H$ or $y^2 = 1_H$, this entails $x^2 = y^2 = 1_H$. Consequently, $\{1_H, x\}\{1_H, xy\}\{1_H, y\} = \{1_H, x, y, xy, x^2y, xy^2, x^2y^2\} = \{1_H, x, y, xy\} = \{1_H, x\}\{1_H, y\}$.
By applying f to the equation, it follows that

$$ab \neq g(xy) \in \{1_K, a\}\{1_K, g(xy)\}\{1_K, b\} = \{1_K, a\}\{1_K, b\} = \{1_K, a, b, ab\}. \tag{7}$$

Now, we gather from the injectivity of g that $xy \in \{1_H, x, y\}$, which, by the cancellativity of H, yields either $xy = 1_H$, or $x = 1_H$, or $y = 1_H$. However, if $x = 1_H$ or $y = 1_H$, then $g(xy) = ab$, which is absurd. So, the only possibility is that $xy = 1_H$, and since $x^2 = 1_H$ (as noted above), we find that $y = x^2y = x(xy) = x1_H = x$. Accordingly, we are guaranteed by Proposition 11 that $g(xy) = g(x^2) = g(x)^2 = g(x)g(y) = ab$, which is again a contradiction and completes the proof. ■

Properties of the pullback

Theorem 14.

Let g be the pullback of an isomorphism f from $\mathcal{P}_{\mathrm{fin},1}(H)$ to $\mathcal{P}_{\mathrm{fin},1}(K)$, where H and K are cancellative monoids. If $x, y \in H$ are torsion elements, then $g(xy) = g(x)g(y)$.

This is the culmination of our work on torsion elements. It definitively proves that for any two torsion elements x, y in a cancellative monoid, the pullback g respects their product: $g(xy) = g(x)g(y)$. The proof assumes the contrary and, through a series of algebraic manipulations using the previous lemmas, arrives at a contradiction.

Main result

We have now shown that the pullback g is a homomorphism on the set of torsion elements. This puts us in a position to prove our main result of the talk (Theorem 15) and thereby show that Question 3 has a positive answer for the class of torsion groups (Corollary 16).

Theorem 15.

Let H *and* K *be cancellative monoids, and suppose that at least one of them is torsion. Assume, in addition, that there exists an isomorphism* f *from* $\mathcal{P}_{\mathrm{fin},1}(H)$ *to* $\mathcal{P}_{\mathrm{fin},1}(K)$. *Then* H *and* K *are both groups, and the pullback of* f *is an isomorphism from* H *to* K.

Corollary 16.

If H *and* K *are groups, at least one of which is torsion, and* $\mathcal{P}_{\mathrm{fin},1}(H)$ *is isomorphic to* $\mathcal{P}_{\mathrm{fin},1}(K)$, *then* H *is isomorphic to* K.

Outline

References

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